PHYSICAL REVIEW E VOLUME 59, NUMBER 6 JUNE 1999

Geometrical and wave optics of paraxial beams

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(Received 30 October 1998)

Most calculational techniques used to evaluate beam propagation are geared towards either fully coherent or fully incoherent beams. The intermediate partial-coherence regime, while in principle known for a long time, has received comparably little attention so far. The resulting shortage of adequate calculational techniques is currently being felt in the realm of x-ray optics where, with the advent of third generation synchrotron light sources, partially coherent beams become increasingly common. The purpose of this paper is to present a calculational approach which, utilizing a ''variance matrix'' representation of paraxial beams, allows for a straightforward evaluation of wave propagation through an optical system. Being capable of dealing with an arbitrary degree of coherence, this approach covers the whole range from wave to ray optics, in a seamless fashion. [S1063-651X(99)03506-0]

PACS number(s): 42.25.Kb

I. INTRODUCTION

The two extreme situations encountered with respect to beam propagation are [1–3] (1) coherent propagation, with a well defined phase relationship across the whole wave front and (2) incoherent propagation, where the phase changes randomly across the wave front, with a transverse coherence length (TLC) much shorter than the transverse dimensions of the beam. The mathematical techniques needed to deal with these extreme situations are well established. In the first, coherent case, standard wave propagation techniques are perfectly adequate. In the second case either ray optics or the more sophisticated geometrical phase space approach may be successfully applied. All these techniques are powerful enough to fully account for beam behavior in the paraxial approximation, within their respective regimes.

Between these two regimes, however, lies the gray area of partial coherence, where TCLs are comparable to the transverse dimensions of the beam and neither of the techniques mentioned above may be relied upon. Some general wave based techniques dealing with partial coherence have been devised by Zernike and others [2–4], more than half a century ago, but, although very general, these approaches are too unwieldy to yield more than qualitative results, in most cases. As for phase space techniques, these are purely geometrical, thus inherently incoherent. Therefore, as far as beam propagation calculations go, the attitude towards the partial coherence region is just to avoid it as far as possible. However, it is not always possible.

In x-ray applications, traditionally, beam coherence was not a significant issue. Thanks to the short wavelengths being used, TCLs were usually orders of magnitude smaller than beam dimensions, thus justifying an incoherent treatment of beam propagation. In fact it took a significant experimental effort to create situations where coherence related effects could have been observed [5,6]. However, with the advent of third generation synchrotron light sources, the above is no longer true. Given the small source sizes and long flight paths used on these machines, TCLs of tens and even hundreds of microns are routine. This is comparable to the typical transverse dimensions of the beams and, even more important, to the dimensions of the apertures which are commonly used to define and delimit the beams. Ignoring coherence properties, in such situations, may yield huge errors in beam size and intensity estimates. As, on the other hand, the beams are rarely coherent enough to justify a pure wave treatment, partial coherence is becoming the norm, rather than the exception, in x-ray optics [7].

The purpose of this paper is to present a ''variance matrix'' based calculational approach which allows us to evaluate the propagation of a paraxial beam through an optical system, in a straightforward manner. Using a propagator derived from the Kirchhoff diffraction integral as a starting point, this approach results in a representation covering the whole range from fully coherent to fully incoherent beams (i.e., from wave optics to ray optics) in a smooth and seamless fashion. A general presentation of this formalism is provided in Sec. II, with the more technical details relegated to Appendix A. Section III covers an application of the formalism to fully and partially coherent Gaussian beams, while beam-aperture interaction is described in Sec. IV. Some important general comments are included in Sec. V. For completeness and comparison purposes a brief summary of the geometrical phase space approach is provided in Appendix B.

II. GENERAL BEAMS

A. Definitions

The following formalism applies to paraxial beams, with a single relevant transverse dimension x, propagating along the z axis. A beam is represented by a wave amplitude, either a space amplitude $\psi(x)$ or an angle amplitude $\tilde{\psi}(\theta)$ (the latter is also referred to as the ''momentum amplitude'' since the angle θ and transverse momentum are equivalent). These amplitudes are related through

$$\psi(x)=\sqrt{\frac{k}{2\pi}}\int \tilde{\psi}(\theta)\exp(ik\theta x)\,d\theta,$$
$$\tilde{\psi}(\theta)=\sqrt{\frac{k}{2\pi}}\int \psi(x)\exp(-ik\theta x)\,dx, \tag{2.1}$$

where $k=2\pi/\lambda$ is the wave vector. The relations (2.1) are formally identical to the position-momentum relations in quantum mechanics (QM) and, similar to QM, position and angle can be represented using the operators

$$x \leftrightarrow -\frac{1}{ik}\frac{d}{d\theta}, \qquad \theta \leftrightarrow \frac{1}{ik}\frac{d}{dx}. \tag{2.2}$$

From Eq. (2.2) follows the equivalent of the QM uncertainty relation, in the form

$$x\theta-\theta x=\frac{1}{ik}\left(x\frac{d}{dx}-\frac{d}{dx}x\right)=-\frac{1}{ik}. \tag{2.3}$$

While the wave amplitudes, as introduced above, are the best representation for coherent beams, partially coherent ones are better represented by a variance matrix **V**, defined by

$$\mathbf{V}=\begin{pmatrix} V_{xx} & V_{x\theta} \\ V_{\theta x} & V_{\theta\theta}\end{pmatrix}=\begin{pmatrix} \langle x^2\rangle & \langle x\theta\rangle \\ \langle \theta x\rangle & \langle \theta^2\rangle\end{pmatrix}, \tag{2.4}$$

where the averages used in Eq. (2.4) are defined (again similar to QM) as follows:

$$\begin{aligned}
\langle x^2\rangle &= \int \psi^* x^2 \psi\, dx = -\frac{1}{k^2}\int \tilde{\psi}^* \frac{d^2}{d\theta^2}\tilde{\psi}\, d\theta \\
&= \frac{1}{k^2}\int \frac{d\tilde{\psi}^*}{d\theta}\frac{d\tilde{\psi}}{d\theta}d\theta, \\
\langle \theta^2\rangle &= \frac{1}{k^2}\int \frac{d\psi^*}{dx}\frac{d\psi}{dx}dx = -\frac{1}{k^2}\int \psi^* \frac{d^2}{dx^2}\psi\, dx \\
&= \int \tilde{\psi}^* \theta^2 \tilde{\psi}\, d\theta, \\
\langle x\theta\rangle &= \frac{1}{ik}\int \psi^* x \frac{d\psi}{dx}dx = -\frac{1}{ik}\int \tilde{\psi}^* \frac{d}{d\theta}\theta\tilde{\psi}\, d\theta \\
&= \frac{1}{ik}\int \tilde{\psi}\theta\frac{d\tilde{\psi}^*}{d\theta}d\theta, \\
\langle \theta x\rangle &= \frac{1}{ik}\int \psi^* \frac{d}{dx}x\psi\, dx = -\frac{1}{ik}\int \psi x\frac{d\psi^*}{dx}dx \\
&= -\frac{1}{ik}\int \tilde{\psi}^*\theta\frac{d\tilde{\psi}}{d\theta}d\theta.
\end{aligned} \tag{2.5}$$

In all the calculations above it is assumed that the space and angle intensities are normalized, i.e.,

$$\int I(x)dx=\int \psi^*(x)\psi(x)dx=1, \qquad \int \tilde{I}(\theta)d\theta=\int \tilde{\psi}^*(\theta)\tilde{\psi}(\theta)d\theta=1 \tag{2.6}$$

or else the integrals in Eq. (2.5) need to be divided by the values of the intensity integrals. Note that the values of $\langle x\theta\rangle$ and $\langle \theta x\rangle$ are defined separately as, due to Eq. (2.3), they are not equal. They are related, though. Defining the symmetrized value

$$\langle x\theta\rangle_R=\langle \theta x\rangle_R=\frac{1}{2}(\langle x\theta\rangle+\langle \theta x\rangle) \tag{2.7}$$

it can be shown that

$$\langle x\theta\rangle=\langle x\theta\rangle_R-\frac{1}{2ik}, \qquad \langle \theta x\rangle=\langle \theta x\rangle_R+\frac{1}{2ik}. \tag{.28}$$

In the following it will be shown that the determinant of **V** is invariant under propagation and focusing. The value of the determinant is

$$\begin{aligned}|\mathbf{V}|&=\langle x^2\rangle\langle\theta^2\rangle-\langle x\theta\rangle\langle\theta x\rangle=\langle x^2\rangle\langle\theta^2\rangle-\langle x\theta\rangle_R^2-\frac{1}{4k^2}\\
&=\varepsilon^2-\frac{1}{4k^2},\end{aligned} \tag{2.9}$$

where [see Eq. (B4)]

$$\varepsilon=[\langle x^2\rangle\langle\theta^2\rangle-\langle x\theta\rangle_R^2]^{1/2} \tag{2.10}$$

is the beam emittance (alternatively the definition $\varepsilon=|\mathbf{V}|$ may also be used). At times, especially when dealing with Gaussian beams, it is convenient to use also the inverse of **V**, given by

$$\mathbf{Q}=\mathbf{V}^{-1}=\frac{1}{\varepsilon^2-\frac{1}{4k^2}}\begin{pmatrix} \langle\theta^2\rangle & -\langle x\theta\rangle_R+\frac{1}{2ik} \\ -\langle x\theta\rangle_R-\frac{1}{2ik} & \langle x^2\rangle\end{pmatrix}. \tag{2.11}$$

Note that when $\varepsilon=\frac{1}{2}k$, **V** is singular and **Q** is undefined.

B. Beam propagation

Based on the Kirchhoff diffraction integral, propagation of a wave front over distance z is described by

$$\psi_z(x)=\int \psi_0(x')\chi(x-x',z)dx', \tag{2.12}$$

where the indices 0 and z denote the initial and final wave, respectively, and (see Appendix A) the propagator χ, in the one-dimensional (1D) case discussed here, is given by

$$\chi(x-x',z)=\left(\frac{-ik}{2\pi z}\right)^{1/2}\exp\left[ik\left(z+\frac{|x-x'|^2}{2z}\right)\right]. \tag{2.13}$$

The angle-representation propagation law is obtained by Fourier transforming Eq. (2.12) [see Appendix A, Eqs. (A21) and (A22)]. The result is

$$\tilde{\psi}_z(\theta)=\exp\left[ikz\left(1-\frac{\theta^2}{2}\right)\right]\tilde{\psi}_0(\theta)=\tilde{\chi}(\theta,z)\tilde{\psi}_0(\theta). \tag{2.14}$$

Note that the angle distribution $\tilde{I}(\theta)$ is invariant under propagation as $\tilde{\psi}_z^*\tilde{\psi}_z=\tilde{\psi}_0^*\tilde{\psi}_0$. Also, differentiation of Eq. (2.14) yields

$$\frac{d\tilde{\psi}_z}{d\theta}=\exp\left[ikz\left(1-\frac{\theta^2}{2}\right)\right]\left(\frac{d\tilde{\psi}_0}{d\theta}-ikz\theta\tilde{\psi}_0\right). \tag{2.15}$$

The angle representation is especially convenient for the evaluation of the transformation of **V** under propagation. Using Eqs. (2.14) and (2.15), averages evaluated at z can be expressed as linear combinations of averages evaluated at 0. The calculations are straightforward and they yield

$$\begin{aligned}\mathbf{V}_z&=\begin{pmatrix}\langle x^2\rangle_z & \langle x\theta\rangle_z\\ \langle\theta x\rangle_z & \langle\theta^2\rangle_z\end{pmatrix}\\ &=\begin{pmatrix}\langle x^2\rangle_0+z(\langle x\theta\rangle_0+\langle\theta x\rangle_0)+z^2\langle\theta^2\rangle_0 & \langle x\theta\rangle_0+z\langle\theta^2\rangle_0\\ \langle\theta x\rangle_0+z\langle\theta^2\rangle_0 & \langle\theta^2\rangle_0\end{pmatrix}\\ &=\mathbf{T}_z\mathbf{V}_0\mathbf{T}_z^T,\end{aligned} \tag{2.16}$$

where, again, the indices 0 and z denote quantities evaluated at the initial and final point, respectively, and the operator **T** is given by

$$\mathbf{T}_z=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}. \tag{2.17}$$

Note that the transformation induced by $\mathbf{T}_z$ is emittance preserving as it does not change the determinant of **V**. These results are identical to the geometrical optics results in Appendix B [see Eqs. (B7) and (B8)]. And, as in Appendix B, the result (2.16) can be used to characterize a beam in terms of ''source'' and distance. A ''source''is a location where $\langle x\theta\rangle_R=0$, i.e., the off-diagonal terms of **V** are purely imaginary. An inspection of Eq. (2.16) shows that such condition occurs upon propagation by

$$z=-z_s=-\frac{\langle x\theta\rangle_R}{\langle\theta^2\rangle}=-\frac{V_{x\theta}+V_{\theta x}}{2V_{\theta\theta}}. \tag{2.18}$$

At the source, **V** obtains the form

$$\mathbf{V}=\begin{pmatrix}V_{xx}-\dfrac{(V_{x\theta}+V_{\theta x})^2}{4V_{\theta\theta}} & -\dfrac{1}{2ik}\\ \dfrac{1}{2ik} & V_{\theta\theta}\end{pmatrix}=\begin{pmatrix}\dfrac{\varepsilon^2}{V_{\theta\theta}} & -\dfrac{1}{2ik}\\ \dfrac{1}{2ik} & V_{\theta\theta}\end{pmatrix}. \tag{2.19}$$

Note that the imaginary off-diagonal terms which characterize a source matrix are proportional to $1/k$. At the geometrical limit of $k\to\infty$, these terms tend to 0, in agreement with the geometrical source definition. Now, introducing the notation

$$\begin{aligned}\sigma_x^2&=V_{xx}-\frac{(V_{x\theta}+V_{\theta x})^2}{4V_{\theta\theta}},\\ \sigma_\theta^2&=V_{\theta\theta},\end{aligned} \tag{2.20}$$

a formal identity is established between the results above and those of geometrical optics [see Eqs. (B9)–(B11)]. Equations (2.18)–(2.20) allow us to describe an arbitrary beam as being propagated, over distance z_s from a source characterized by σ_x,σ_θ (i.e., the source rms. values of x and θ). Note that Eqs. (2.19) and (2.20) yield $\sigma_x\sigma_\theta=\varepsilon$. This can serve as an alternative definition of emittance.

C. Beam focusing

The action of a focusing device with focal length F is best described in the space representation, through

$$\psi_F(x)=\exp\left(-\frac{ikx^2}{2F}\right)\psi_0(x), \tag{2.21}$$

where the indices 0 and F denote the state before and after the focusing, respectively. Differentiation of Eq. (2.21) yields

$$\frac{d\psi_F}{dx}=\exp\left(-\frac{ikx^2}{2F}\right)\left(\frac{d\psi_0}{dx}-\frac{ikx}{F}\psi_0\right). \tag{2.22}$$

As in the case of propagation, Eqs. (2.21) and (2.22) allow us to express the averages appearing in the components of **V**, after the focusing is applied, in terms of the original components. The calculations (in the space representation, this time) yield

$$\begin{aligned}\mathbf{V}_F=\begin{pmatrix}\langle x^2\rangle_F & \langle x\theta\rangle_F\\ \langle\theta x\rangle_F & \langle\theta^2\rangle_F\end{pmatrix}&=\begin{pmatrix}\langle x^2\rangle_0 & \langle x\theta\rangle_0-\dfrac{1}{F}\langle x^2\rangle_0\\ \langle\theta x\rangle_0-\dfrac{1}{F}\langle x^2\rangle_0 & \langle\theta^2\rangle_0-\dfrac{1}{F}(\langle x\theta\rangle_0+\langle\theta x\rangle_0)+\dfrac{1}{F^2}\langle x^2\rangle_0\end{pmatrix},\\ &=\mathbf{T}_F\mathbf{V}_0\mathbf{T}_F{}^T,\end{aligned} \tag{2.23}$$

where

$$\mathbf{T}_F=\begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}. \tag{2.24}$$

As before, the determinant of $\mathbf{V}$ remains unchanged by the transformation. The results (2.23) and (2.24) are identical to the geometric optics results in Eqs. (B12) and (B13). Applying Eqs. (2.18) and (2.20) to the results above one can find the new ''source'' parameters, generated by the focusing. Expressed in terms of the original parameters the results are

$$\begin{aligned} \sigma'^2_x &= \frac{F^2\sigma_x^2\sigma_\theta^2}{\sigma_x^2+(F-z_s)^2\sigma_\theta^2}, \\ \sigma'^2_\theta &= \frac{\sigma_x^2+(F-z_s)^2\sigma_\theta^2}{F^2}, \\ z'_s &= \frac{F[z_s(F-z_s)\sigma_\theta^2-\sigma_x^2]}{\sigma_x^2+(F-z_s)^2\sigma_\theta^2}, \end{aligned} \tag{2.25}$$

again the same as the geometrical optics results.

D. Diffuser

A diffuser is a device or a physical process introducing random phase shifts in the wave front. It can be mathematically represented by

$$\psi_D(x)=\exp\left(\frac{i\varphi(x)}{2}\right)\psi_0(x), \tag{2.26}$$

where the indices 0 and D refer to the state before and after the application of the diffuser. Differentiation of Eq. (2.26) yields

$$\frac{d\psi_D}{dx}=\exp\left(\frac{i\varphi(x)}{2}\right)\left(\frac{d\psi_0}{dx}+\frac{i}{2}\,\frac{d\varphi}{dx}\,\psi_0\right). \tag{2.27}$$

The $\varphi(x)$, appearing in Eqs. (2.26) and (2.27), is a random phase (the $\frac{1}{2}$ factor is used for calculational convenience and carries no physical significance). In the following, averaging over random factors is always performed. This averaging, denoted by a bar over the averaged quantity, is to be understood as averaging over a set of similar diffusers or, if φ is time dependent, as time averaging. The ''bar'' averaging is performed before the ''angle bracket'' averaging and, for bar averaging purposes, φ is assumed to fulfill the relations

$$\begin{aligned} \overline{\frac{d\varphi}{dx}} &= 0, \\ \overline{\left(\frac{d\varphi}{dx}\right)^2} &= \frac{1}{\Gamma_D^2}, \end{aligned} \tag{2.28}$$

where Γ_D is a constant (for a given diffuser) parameter, with the dimension of length, which serves as a measure of ''correlation length.'' It will be shown later that Γ_D is related to the transverse coherence length of the beam.

Now, with the aid of Eqs. (2.26)–(2.28) we can evaluate the matrix $\mathbf{V}$ or, more exactly $\overline{\mathbf{V}}$, since bar averaging needs to be performed on all the elements. However [see Eq. (2.5)], the exponential phase factor cancels in all the calculations and derivatives of φ are eliminated by the bar averaging. Therefore $\langle x^2\rangle$, as well as $\langle x\theta\rangle$ and $\langle \theta x\rangle$ remain unchanged. In $\langle \theta^2\rangle$, on the other hand, there are terms proportional to the square of $d\varphi/dx$ [see, again, Eq. (2.5)] and these terms yield a nonzero contribution. We get

$$\begin{aligned} \langle\overline{\theta^2}\rangle_D &= \frac{1}{k^2}\int\left[\frac{d\psi_0^*}{dx}\,\frac{d\psi_0}{dx}+\frac{i}{2}\,\overline{\frac{d\varphi}{dx}}\left(\frac{d\psi_0^*}{dx}\,\psi_0-\psi_0^*\frac{d\psi_0}{dx}\right)\right. \\ &\quad \left.+\frac{1}{4}\overline{\left(\frac{d\varphi}{dx}\right)^2}\,\psi_0^*\psi_0\right]dx \\ &= \frac{1}{k^2}\int\frac{d\psi_0^*}{dx}\,\frac{d\psi_0}{dx}\,dx+\frac{1}{4k^2\Gamma_D^2}\int\psi_0^*\psi_0\,dx \\ &= \langle\theta^2\rangle_0+\frac{1}{4k^2\Gamma_D^2}. \end{aligned} \tag{2.29}$$

Therefore

$$\overline{\mathbf{V}_D}=\begin{pmatrix} \langle x^2\rangle_0 & \langle x\theta\rangle_0 \\ \langle\theta x\rangle_0 & \langle\theta^2\rangle_0+\frac{1}{4k^2\Gamma_D^2} \end{pmatrix}. \tag{2.30}$$

Note that, unlike in the case of propagation and focusing, the emittance does change here as

$$|\overline{\mathbf{V}_D}|=|\mathbf{V}_0|+\frac{\langle x^2\rangle}{4k^2\Gamma_D^2}. \tag{2.31}$$

Applying Eqs. (2.18)–(2.20) to the result (2.30), the new beam parameters are obtained. Expressed in terms of the original parameters the results are

$$\begin{aligned} \sigma'^2_x &= \sigma_x^2+\frac{z_s^2\sigma_\theta^2}{1+4k^2\Gamma_D^2\sigma_\theta^2}, \\ \sigma'^2_\theta &= \sigma_\theta^2+\frac{1}{4k^2\Gamma_D^2}, \\ z'_s &= \frac{4k^2\Gamma_D^2\sigma_\theta^2}{1+4k^2\Gamma_D^2\sigma_\theta^2}\,z_s. \end{aligned} \tag{2.32}$$

It is worth noting that the results (2.29)–(2.32) are wave optics results, having no geometrical counterparts. This is evidenced by the fact that the changes induced by the diffuser vanish in the geometrical optics limit, being proportional to $1/k^2$.

Once the random phase has been introduced (either by the source or by roughness in the optical devices) it becomes a property of the beam itself, separate from the device which caused it. When viewed as such, the correlation length will be referred to as Γ, without a subscript. Now, it is to be expected that the presence of random phase in one location along the beam propagation path implies the presence of random phase in all following locations, possibly with different Γ (which, thus, becomes a function of z). According to Eq. (2.31), the device responsible for the randomness contributes $\langle x^2\rangle/4k^2\Gamma_D^2$ to the emittance. But, as the beam propagates onward, the emittance remains constant. Therefore the relation

$$\frac{\langle x^2(z_1)\rangle}{\Gamma^2(z_1)}=\frac{\langle x^2(z_2)\rangle}{\Gamma^2(z_2)} \tag{2.33}$$

must be true for arbitrary z_1, z_2. Especially, since $\langle x^2(z)\rangle=\sigma_x^2+z^2\sigma_\theta^2$ [see Eq. (2.16)], Γ at any location z can be related to Γ at the source through

$$\Gamma^2(z)=\Gamma^2(0)\,\frac{\langle x^2\rangle_z}{\langle x^2\rangle_0}=\Gamma^2(0)\left(1+\frac{\sigma_\theta^2 z^2}{\sigma_x^2}\right). \tag{2.34}$$

In the following sections the bar average of the expression $\exp\{i[\varphi(x')-\varphi(x'')]/2\}$ will often be used. Due to the randomness of the phase this expression can be expected to average to zero, except where x' is very close to x''. This justifies the approximation

$$\begin{aligned}\overline{\exp\left(\frac{i}{2}[\varphi(x')-\varphi(x')]\right)}&\approx\overline{\exp\left(\frac{i}{2}(x'-x'')\frac{d\varphi}{dx}\bigg|_{x=x'}\right)}\\&\approx\overline{1+\frac{i}{2}(x'-x'')\frac{d\varphi}{dx}-\frac{1}{8}(x'-x'')^2\left(\frac{d\varphi}{dx}\right)^2}\\&=1-\frac{(x'-x'')^2}{8\Gamma^2}\approx\exp\left(-\frac{(x'-x'')^2}{8\Gamma^2}\right).\end{aligned} \tag{2.35}$$

III. GAUSSIAN BEAMS

A. Coherent beams

In order to proceed beyond the general results of the previous section, a specific beam profile needs to be assumed. Gaussian beam profiles, due to the fact that the Gaussian form is maintained under propagation and focusing, are especially convenient. The simplest (normalized) Gaussian wave amplitude is given by

$$\psi(x)=\frac{1}{(2\pi\sigma_x^2)^{1/4}}\exp\left(-\frac{x^2}{4\sigma_x^2}\right), \tag{3.1}$$

yielding an intensity of

$$I(x)=\frac{1}{(2\pi\sigma_x^2)^{1/2}}\exp\left(-\frac{x^2}{2\sigma_x^2}\right). \tag{3.2}$$

Using Eq. (2.5) to evaluate all the needed averages, the corresponding variance matrix can be constructed. The result is

$$\mathbf{V}=\begin{pmatrix}\sigma_x^2 & -\dfrac{1}{2ik}\\ \dfrac{1}{2ik} & \dfrac{1}{4k^2\sigma_x^2}\end{pmatrix}=\begin{pmatrix}\sigma_x^2 & -\dfrac{1}{2ik}\\ \dfrac{1}{2ik} & \sigma_\theta^2\end{pmatrix}. \tag{3.3}$$

Note that, using the definitions of the previous section, this $\mathbf{V}$ represents a source, as the off diagonal elements are imaginary. Note also that $\sigma_\theta=\frac{1}{2}k\sigma_x$, thus σ_θ is not an independent quantity here. The emittance of the source is

$$\varepsilon=\sigma_x\sigma_\theta=\sigma_x\frac{1}{2k\sigma_x}=\frac{1}{2k}, \tag{3.4}$$

which, as mentioned in Sec. II A is the minimal emittance possible. The source represented by Eq. (3.1) is fully coherent and its $\mathbf{V}$ matrix satisfies $|\mathbf{V}|=0$, thus it is not invertible. It will be shown later that once some amount of incoherence is introduced, this singularity disappears.

The angle amplitude is obtained applying a Fourier transform to Eq. (3.1) [see definitions (2.1)], resulting in

$$\begin{aligned}\tilde{\psi}(\theta)&=\left(\frac{4k^2\sigma_x^2}{2\pi}\right)^{1/4}\exp(-ik^2\sigma_x^2\theta^2)\\&=\frac{1}{(2\pi\sigma_\theta^2)^{1/4}}\exp\left(-\frac{\theta^2}{4\sigma_\theta^2}\right),\end{aligned} \tag{3.5}$$

where the relation $\sigma_\theta=\frac{1}{2}k\sigma_x$ was used. This is of the same form as Eq. (3.1). Also, the angle intensity is of same form as Eq. (3.2), being given by

$$\tilde{I}(\theta)=\frac{1}{(2\pi\sigma_\theta^2)^{1/2}}\exp\left(-\frac{\theta^2}{2\sigma_\theta^2}\right). \tag{3.6}$$

The transformation induced in $\mathbf{V}$ under propagation by distance z is evaluated applying Eq. (2.16) to Eq. (3.3), resulting in

$$\mathbf{V}_z=\begin{pmatrix}\sigma_x^2+z^2\sigma_\theta^2 & z\sigma_\theta^2-\dfrac{1}{2ik}\\ z\sigma_\theta^2+\dfrac{1}{2ik} & \sigma_\theta^2\end{pmatrix}. \tag{3.7}$$

However, in this case one can actually calculate the propagated amplitude itself, not just the variance matrix. Using Eqs. (2.12) and (2.13) we get

$$\begin{aligned}\psi_z(x)&=\frac{1}{(2\pi\sigma_x^2)^{1/4}}\left(\frac{-ik}{2\pi z}\right)^{1/2}\exp(ikz)\\&\quad\times\int\exp\left(-\frac{x'^2}{4\sigma_x^2}+ik\frac{(x-x')^2}{2z}\right)dx'\\&=\left[\exp(ikz)\,\frac{(\sigma_x-iz/2k\sigma_x)^{1/2}}{(\sigma_x^2+z^2/4k^2\sigma_x^2)^{1/4}}\right]\\&\quad\times\frac{1}{[2\pi(\sigma_x^2+z^2/4k^2\sigma_x^2)]^{1/4}}\\&\quad\times\exp\left(-\frac{x^2(1-iz/2k\sigma_x^2)}{4(\sigma_x^2+z^2/4k^2\sigma_x^2)}\right).\end{aligned} \tag{3.8}$$

But the first factor (in square brackets) in this result can be ignored, being just an x-independent phase factor. As for the reminder, using the substitution [see Eq. (3.7)]

$$\sigma_x^2+\frac{z^2}{4k^2\sigma_x^2}=\sigma_x^2+z^2\sigma_\theta^2=\langle x^2\rangle, \tag{3.9}$$

it can be rewritten as

$$\psi_z(x)=\frac{1}{(2\pi\langle x^2\rangle)^{1/4}}\exp\left(-\frac{x^2(1-2ik\sigma_\theta^2 z)}{4\langle x^2\rangle}\right), \tag{3.10}$$

yielding

$$I_z(x)=\psi_z^*(x)\psi_z(x)=\frac{1}{(2\pi\langle x^2\rangle)^{1/2}}\exp\left(-\frac{x^2}{2\langle x^2\rangle}\right). \tag{3.11}$$

Thus, the Gaussian form is maintained. As for the propagation of the angle amplitude, applying (2.14) to Eq. (3.5) yields

$$\tilde{\psi}_z(\theta)=\frac{\exp(ikz)}{(2\pi\sigma_\theta^2)^{1/4}}\exp\left(-\frac{\theta^2(1+2ik\sigma_\theta^2 z)}{4\sigma_\theta^2}\right) \tag{3.12}$$

and

$$\tilde{I}_z(\theta)=\tilde{\psi}_z^*(\theta)\tilde{\psi}_z(\theta)=\frac{1}{(2\pi\sigma_\theta^2)^{1/2}}\exp\left(-\frac{\theta^2}{2\sigma_\theta^2}\right). \tag{3.13}$$

Taking into account that $\sigma_\theta^2=\langle\theta^2\rangle$, Eqs. (3.12) and (3.13) are formally identical to Eqs. (3.10) and (3.11)

B. Partially coherent beams

The source in Eq. (3.1) may be modified by an application of a ''diffuser'' (see Sec. II D) with a correlation length Γ, yielding

$$\psi(x)=\frac{1}{(2\pi\sigma_x^2)^{1/4}}\exp\left(-\frac{x^2}{4\sigma_x^2}+i\frac{\varphi(x)}{2}\right), \tag{3.14}$$

where

$$\overline{\left(\frac{d\varphi}{dx}\right)^2}=\frac{1}{\Gamma^2}. \tag{3.15}$$

The presence of the random phase φ means that all the following results are obtained using ''bar'' averaging. Thus, strictly speaking, a bar should appear above all the results. For the sake of notational brevity this will be dispensed with.

The intensity $I(x)$ at the source is not changed by the presence of φ. The variance matrix $\mathbf{V}$, on the other hand, changes. Using Eqs. (2.30) and (3.3), we get

$$\mathbf{V}=\begin{pmatrix}\sigma_x^2 & -\frac{1}{2ik}\\ \frac{1}{2ik} & \frac{1}{4k^2}\left(\frac{1}{\sigma_x^2}+\frac{1}{\Gamma^2}\right)\end{pmatrix}=\begin{pmatrix}\sigma_x^2 & -\frac{1}{2ik}\\ \frac{1}{2ik} & \sigma_\theta^2\end{pmatrix} \tag{3.16}$$

and

$$|\mathbf{V}|=\frac{\sigma_x^2}{4k^2\Gamma^2}. \tag{3.17}$$

Thus the random phase increases both the emittance (note that $\mathbf{V}$ is not singular anymore) and the angular divergence which is now given by

$$\sigma_\theta^2=\frac{1}{4k^2}\left(\frac{1}{\sigma_x^2}+\frac{1}{\Gamma^2}\right). \tag{3.18}$$

Due to the randomness of φ, a closed form calculation of $\tilde{\Psi}(\theta)$ is now impossible. $\tilde{I}(\theta)$, on the other hand, can be calculated, being given by

$$\tilde{I}(\theta)=\overline{\tilde{\psi}^*(\theta)\tilde{\psi}(\theta)}=\frac{k}{2\pi}\frac{1}{\sqrt{2\pi\sigma_x^2}}\int\overline{\exp\left(-\frac{x'^2+x''^2}{4\sigma_x^2}+\frac{i}{2}[\varphi(x')-\varphi(x'')]-ik\theta(x'-x'')\right)}dx'dx''. \tag{3.19}$$

Using Eq. (2.35) and the substitution (A13), Eq. (3.19) can be converted to the double Gaussian integral

$$\tilde{I}(\theta)=\frac{k}{2\pi}\frac{1}{\sqrt{2\pi\sigma_x^2}}\int\times\exp\left[-\left(\frac{u^2}{2\sigma_x^2}+\frac{v^2}{8\sigma_x^2}+\frac{v^2}{8\Gamma^2}-ik\theta v\right)\right]du\,dv. \tag{3.20}$$

Performing the integration and using $1/\sigma_x^2+1/\Gamma^2=4k^2\sigma_\theta^2$ [see Eq. (3.18)], the result

$$\tilde{I}(\theta)=\frac{1}{(2\pi\sigma_\theta^2)^{1/2}}\exp\left(-\frac{\theta^2}{2\sigma_\theta^2}\right), \tag{3.21}$$

is obtained, as in Eq. (3.13), though with a different σ_θ.

The situation with beam propagation is similar. $\mathbf{V}_z$ can be easily evaluated, yielding again the result (3.7). A closed form evaluation of $\psi_z(x)$ is not possible but a calculation of $I_z(x)$ is doable. Using the propagator (2.13) and the fact that $I_z(x)=\overline{\psi_z^*(x)\psi_z(x)}$, the propagated intensity is obtained as

$$I_z(x)=\frac{k}{2\pi z}\frac{1}{\sqrt{2\pi\sigma_x^2}}\int \overline{\exp\left(ik\frac{(x-x')^2-(x-x'')^2}{2z}\right)\exp\left(-\frac{x'^2+x''^2}{4\sigma_x^2}+\frac{i}{2}[\varphi(x')-\varphi(x'')]\right)}dx'dx''. \quad (3.22)$$

Applying, again, the substitution (A13), Eq. (2.35), and Eq. (3.18), Eq. (3.22) transforms into

$$I_z(x)=\frac{k}{2\pi z}\frac{1}{\sqrt{2\pi\sigma_x^2}}\int \times\exp\left[-\left(\frac{u^2}{2\sigma_x^2}+\frac{k^2\sigma_\theta^2\nu^2}{2}+\frac{ik\nu(u-x)}{z}\right)\right]du\,d\nu. \quad (3.23)$$

Integration of Eq. (3.23), using the relation $\langle x^2\rangle=\sigma_x^2+z^2\sigma_\theta^2$ from Eq. (3.7) which is still valid here, yields

$$I_z(x)=\frac{1}{\sqrt{2\pi\langle x^2\rangle}}\exp\left(-\frac{x^2}{2\langle x^2\rangle}\right), \quad (3.24)$$

which has the same form as Eq. (3.11). As for $\tilde{I}_z(\theta)$, it is invariant under propagation as mentioned in Sec. II B. Thus it remains the same as in Eq. (3.21), i.e., of the same form as in Eq. (3.13).

While $\psi_z(x)$ cannot be evaluated directly in the presence of the random phase, one can try, by analogy with Eq. (3.10), to guess a result of the form

$$\psi_z(x)=\frac{1}{(2\pi\langle x^2\rangle)^{1/4}}\exp\left(-\frac{x^2(1-2ik\sigma_\theta^2 z)}{4\langle x^2\rangle}+i\frac{\varphi(x,z)}{2}\right), \quad (3.25)$$

where

$$\overline{\left(\frac{d\varphi(x,z)}{dx}\right)^2}=\frac{1}{\Gamma_z^2}. \quad (3.26)$$

Here Γ_z is a z-dependent correlation length (see Sec. II D). The functional form of this z dependence can be established by comparing the elements of $\mathbf{V}$, as calculated using the function (3.25), with the direct result (3.16). Now, as was discussed in Sec. II D, only the $\langle\theta^2\rangle$ element is influenced by the presence of φ. The calculation is similar to Eq. (2.29) and it yields

$$\langle\theta^2\rangle=\frac{1}{4k^2}\left(\frac{1+4k^2z^2\sigma_\theta^2}{\langle x^2\rangle}+\frac{1}{\Gamma_z^2}\right)=\sigma_\theta^2, \quad (3.27)$$

where the second equality is forced by the necessity of an agreement with Eq. (3.16). Reorganization of the terms yields

$$\frac{1}{\Gamma_z^2}=\frac{4k^2\sigma_\theta^2(\langle x^2\rangle-z^2\sigma_\theta^2)-1}{\langle x^2\rangle}=\frac{(1/\sigma_x^2+1/\Gamma^2)\sigma_x^2-1}{\langle x^2\rangle}=\frac{\sigma_x^2}{\Gamma^2\langle x^2\rangle}. \quad (3.28)$$

Thus

$$\frac{\Gamma_z^2}{\langle x^2\rangle}=\frac{\Gamma^2}{\sigma_x^2}, \quad (3.29)$$

in full agreement with Eq. (2.33). At this stage a new, z-dependent length parameter Λ_z may be introduced, defined through

$$\frac{1}{\Lambda_z^2}=\frac{1}{\langle x^2\rangle}+\frac{1}{\Gamma_z^2}=\frac{\sigma_x^2}{\langle x^2\rangle}\left(\frac{1}{\sigma_x^2}+\frac{1}{\Gamma^2}\right). \quad (3.30)$$

Using Eq. (3.18) and the definition of the beam emittance ε, Eq. (3.30) can be rewritten as

$$\Lambda_z=\frac{\langle x^2\rangle^{1/2}}{2k\sigma_x\sigma_\theta}=\frac{\langle x^2\rangle^{1/2}}{2k\varepsilon}. \quad (3.31)$$

For large values of z, $\langle x^2\rangle\approx z^2\sigma_\theta^2$, thus, far enough away from source Eq. (3.31) results in

$$\Lambda_z\approx\frac{z}{2k\sigma_x}=\frac{\lambda z}{4\pi\sigma_x}\rightarrow\sqrt{4\pi}\Lambda_z\approx\frac{\lambda z}{\sqrt{4\pi}\sigma_x}. \quad (3.32)$$

But the expression on the right-hand side of Eq. (3.32) is just the transverse coherence length of the beam, at distance z from the source, with $\sqrt{4\pi}\sigma_x$ being the effective source size. Therefore we can also identify $\sqrt{4\pi}\Lambda_z$ as the transverse coherence length of the beam. Since Λ_z depends on Γ_z [see Eq. (3.30)] which, in turn, is proportional to the correlation length Γ [see Eq. (3.29)] the relationship between Γ and the transverse coherence length of the beam has therefore been established.

IV. APERTURES

A. Position aperture

The effect of apertures on the beam can be conveniently evaluated using Gaussian apertures. Specifically, the effect of a Gaussian position aperture on the amplitude is represented by

$$\psi_A(x)=\exp\left(-\frac{x^2}{4d^2}\right)\psi(x), \quad (4.1)$$

which amounts to multiplying the intensity by $\exp(-x^2/2d^2)$. The parameter d is a measure of the aperture width (standard aperture of width w may be approximated by a Gaussian one by setting $d=w/\sqrt{4\pi}$). All the averages calculated here are of the form of Eq. (A11), i.e.,

$$\langle O\rangle=N\int \psi_0^*(x'')\{O\}_d\psi_0(x')dx'dx'', \quad (4.2)$$

where O is the operator being calculated and $\{O\}_d$ results from an integration of O with the propagators and the aperture functions [see Eq. (A17)]. The normalization constant N is needed since $\psi_A(x)$ from Eq. (4.1) is not normalized. As the base function $\psi_0(x)$, the Gaussian amplitude (3.14) is used, with bar averaging to account for the random phases. Using the substitution (A13) and Eq. (2.35) yields

$$\begin{aligned}\overline{\psi_0^*(x'')\psi_0(x')} &= \frac{1}{\sqrt{2\pi\sigma_x^2}}\,\overline{\exp\left(-\frac{x'^2+x''^2}{4\sigma_x^2}+\frac{i}{2}[\varphi(x')-\varphi(x'')]\right)}\\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left[-\left(\frac{u^2}{2\sigma_x^2}+\frac{\nu^2}{8\sigma_x^2}+\frac{\nu^2}{8\Gamma^2}\right)\right]\\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left[-\frac{1}{2}\left(\frac{u^2}{\sigma_x^2}+k^2\sigma_\theta^2\nu^2\right)\right]. \end{aligned}\tag{4.3}$$

The result (4.3) and the values of the brackets in Eqs. (A19) and (A20) are all that is needed to calculate the elements of the variance matrix $\mathbf{V}_A$ resulting from the aperture application. First, though, the normalization constant N needs to be evaluated. Using the definition from Appendix A, with the result of Eq. (4.3) and $\{1\}_d$ supplied by Eq. (A20), yields

$$\begin{aligned} N &= \left(\int \overline{\psi_0^*(x'')\psi_0(x')}\{1\}_d dx'\,dx''\right)^{-1}\\ &= \left[\left(\frac{k^2d^2}{2\pi z^2}\right)^{1/2}\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/2}\int\right.\\ &\quad\left.\times\exp\left(-\frac{k^2\nu^2(d^2+z^2\sigma_\theta^2)}{2z^2}-\frac{u^2}{2\sigma_x^2}-\frac{iku\nu}{z}\right)du\,d\nu\right]^{-1}. \end{aligned}\tag{4.4}$$

The integral in Eq. (4.4) is of the standard double-Gaussian variety and, using $\langle x^2\rangle=\sigma_x^2+z^2\sigma_\theta^2$, the final result is simply

$$N=\left(1+\frac{\langle x^2\rangle}{d^2}\right)^{1/2}. \tag{4.5}$$

The remaining integrations involved in calculating the elements of $\mathbf{V}_A$ are of the same type and will be skipped for brevity. The resulting $\mathbf{V}_A$ is given by

$$\mathbf{V}_A=\frac{d^2}{d^2+\langle x^2\rangle}\times\begin{pmatrix}\langle x^2\rangle & \langle x\theta\rangle_R-\dfrac{d^2+\langle x^2\rangle}{2ikd^2}\\ \langle\theta x\rangle_R+\dfrac{d^2+\langle x^2\rangle}{2ikd^2} & \langle\theta^2\rangle+\dfrac{\varepsilon^2}{d^2}+\dfrac{d^2+\langle x^2\rangle}{4k^2d^4}\end{pmatrix}, \tag{4.6}$$

where $\langle\theta^2\rangle=\sigma_\theta^2$ and $\varepsilon=\sigma_x\sigma_\theta$ is the emittance before the aperture. Comparison with the geometrical results (B21) shows that the only difference is in terms proportional to $1/k$ and $1/k^2$ which tend to zero at the geometrical limit of $k\to\infty$.

Evaluation of the determinant of $\mathbf{V}_A$ yields, after some algebra,

$$|\mathbf{V}_A|=\frac{d^2}{d^2+\langle x^2\rangle}\left(\varepsilon^2-\frac{1}{4k^2}\right)=\frac{d^2}{d^2+\langle x^2\rangle}|\mathbf{V}|, \tag{4.7}$$

which is the same as in the geometric case [see Eq. (B22)]. Following from Eq. (4.7), the new emittance is obtained as

$$\varepsilon'^2=|\mathbf{V}_A|+\frac{1}{4k^2}=\frac{d^2}{d^2+\langle x^2\rangle}\left(\varepsilon^2+\frac{\langle x^2\rangle}{4k^2d^2}\right). \tag{4.8}$$

Using Eq. (4.8) it can easily be shown that $\varepsilon\geqslant\frac{1}{2}k\to\varepsilon'\geqslant\frac{1}{2}k$, regardless of the aperture size. Thus $\frac{1}{2}k$ is an absolute lower bound on emittance, in agreement with Sec. II A.

B. Angle (momentum) aperture

The case of an angle aperture is very similar to the one discussed above. The action of such aperture is described, in the angle representation, by

$$\tilde{\psi}_A(\theta)=\exp\left(-\frac{\theta^2}{4\delta^2}\right)\tilde{\psi}(\theta), \tag{4.9}$$

so that the angle intensity is multiplied by $\exp(-\theta^2/2\delta^2)$. Here δ is a measure of the aperture's angular width. The calculations are slightly different than the previous case since, even though the aperture action occurs in angle space, the Gaussian amplitude (3.14), in position space, is still used as the base function. Therefore the mixed propagators from Eq. (A24) are used in the preliminary averaging and $\{\}_\delta$ replaces $\{\}_d$ throughout the calculations. Specifically, the calculation of the normalization constant N utilizes $\{1\}_\delta$ from Eq. (A30), resulting in

$$\begin{aligned} N &= \left(\int \overline{\psi_0^*(x'')\psi_0(x')}\{1\}_\delta dx'\,dx''\right)^{-1}\\ &= \left[\left(\frac{k^2d^2}{2\pi z^2}\right)^{1/2}\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/2}\int\right.\\ &\quad\left.\times\exp\left(-\frac{k^2\nu^2(\delta^2+\sigma_\theta^2)}{2}-\frac{u^2}{2\sigma_x^2}\right)du\,d\nu\right]^{-1}, \end{aligned}\tag{4.10}$$

which yields

$$N=\left(1+\frac{\sigma_\theta^2}{\delta^2}\right)^{1/2}=\left(1+\frac{\langle\theta^2\rangle}{\delta^2}\right)^{1/2}. \tag{4.11}$$

Using the value of N, the result (4.3) and the brackets from Eqs. (A29) and (A30), all the elements of $\mathbf{V}_A$ may be evaluated according to Eq. (A25). The result is

TABLE I. Aperture modified source parameters and wave results.

Parameter	Position aperture	Angle (momentum) aperture
σ'^2_x	$\dfrac{d^2[\sigma_x^2+\eta_d^2(\sigma_x^2+z_s^2\sigma_\theta^2)]}{d^2+\sigma_x^2+\eta_d^2(d^2+\sigma_x^2+z_s^2\sigma_\theta^2)}$	$(1+\eta_\delta^2)\sigma_x^2$
σ'^2_θ	$\left(\dfrac{d^2+\sigma_x^2}{d^2+\sigma_x^2+z_s^2\sigma_\theta^2}+\eta_d^2\right)\sigma_\theta^2$	$\dfrac{\delta^2\sigma_\theta^2}{\delta^2+\sigma_\theta^2}$
z'_s	$\dfrac{d^2 z_s}{d^2+\sigma_x^2+\eta_d^2(d^2+\sigma_x^2+z_s^2\sigma_\theta^2)}$	z_s

TABLE II. Aperture modified source parameters and geometrical results.

Parameter	Position aperture	Angle (momentum) aperture
σ'^2_x	$\dfrac{d^2\sigma_x^2}{d^2+\sigma_x^2}$	σ_x^2
σ'^2_θ	$\dfrac{(d^2+\sigma_x^2)\sigma_\theta^2}{d^2+\sigma_x^2+z_s^2\sigma_\theta^2}$	$\dfrac{\delta^2\sigma_\theta^2}{\delta^2+\sigma_\theta^2}$
z'_s	$\dfrac{d^2 z_s}{d^2+\sigma_x^2}$	z_s

$$\mathbf{V}_A=\frac{\delta^2}{\delta^2+\langle\theta^2\rangle}\times\begin{pmatrix}\langle x^2\rangle+\dfrac{\varepsilon^2}{\delta^2}+\dfrac{\delta^2+\langle\theta^2\rangle}{4k^2\delta^4} & \langle x\theta\rangle_R-\dfrac{\delta^2+\langle\theta^2\rangle}{2ik\delta^2}\\ \langle\theta x\rangle_R\dfrac{\delta^2+\langle\theta^2\rangle}{2ik\delta^2} & \langle\theta^2\rangle\end{pmatrix}, \tag{4.12}$$

where, again, ε is the emittance before the aperture. Note that Eq. (4.12) could have been obtained straight from Eq. (4.6), by switching the roles of x and θ (and replacing d with δ). Comparison with the geometrical result (B23) shows, again, that apart from terms which go to 0 at the geometrical limit, the results are identical. Evaluation of determinant of $\mathbf{V}_A$ yields

$$|\mathbf{V}_A|=\frac{\delta^2}{\delta^2+\langle\theta^2\rangle}\left(\varepsilon^2-\frac{1}{4k^2}\right)=\frac{\delta^2}{\delta^2+\langle\theta^2\rangle}|\mathbf{V}| \tag{4.13}$$

and, therefore

$$\varepsilon'^2=|\mathbf{V}_A|+\frac{1}{4k^2}=\frac{\delta^2}{\delta^2+\langle\theta^2\rangle}\left(\varepsilon^2+\frac{\langle\theta^2\rangle}{4k^2\delta^2}\right). \tag{4.14}$$

This is formally identical to Eq. (4.8) and the comment at the end of Sec. IV A applies here too.

C. Modified source parameters

The modified source parameters, following the apertures, can be evaluated applying Eqs. (2.18) and (2.20) to the $\mathbf{V}_A$ matrices in Eqs. (4.6) and (4.12). The results are listed in Table I, expressed in terms of the initial values and of the dimensionless parameters

$$\eta_d=\left.\frac{1}{2k\,d\sigma_\theta}\right| \text{for position aperture,}$$
$$\eta_\delta=\left.\frac{1}{2k\delta\sigma_x}\right| \text{for angle aperture.} \tag{4.15}$$

The magnitude of the η parameters provides a measure of the importance of wave effects, for a given aperture. Note that both values tend to 0 at the geometrical limit and that, at the same limit, the results in Table I converge to those of Table II as calculated in Appendix B.

V. SUMMARY AND CONCLUSIONS

As has been shown, the formalism described in the preceding sections can deal with the important aspects of beam propagation, focusing and interaction with apertures, covering in a smooth, seamless fashion the whole range from purely coherent wave description to the purely incoherent geometric phase space one. It is, of course, limited to paraxial beams since, as was mentioned in Sec. II and in Appendix A, only terms up to second order in $(x-x')$ are maintained in the phase of the propagator. This is a generous limitation, though, sufficient to cover both the Fraunhofer and the Fresnel ranges.

It can be asked how much farther can this formalism be generalized and what other optical elements may be modeled using this description? To answer this one should note that, in view of the discussion in Secs. II–IV, the distinction between beam profile and optical elements is somewhat artificial. An arbitrary beam profile may be viewed as the result of an application of a set of optical elements to a plane wave which (as far as x dependence is considered) is represented by a constant. The Gaussian profile in Eq. (3.14), for example, results from an application of a Gaussian aperture and a diffuser to a plane wave. Therefore the question above can be rephrased as ''what is the most general beam profile that needs to be considered''?

Since the propagator phase has been limited to terms up to second order in $(x-x')$, the most general continuous profile conceivable is of the form of $\exp[P(x)]$ where P is a quadratic polynomial, i.e., $P=a+bx+cx^2$ (the discussion here is in the space representation, it can be performed in the angle representation as well, yielding same results). Now, the constant term in P is of no consequence as it only influences the normalization. As for the linear term, its real part can be eliminated by shifting the origin of x. The imaginary part corresponds (as can be readily verified) to a redirection of the beam along an axis making a constant angle with the z axis and can be eliminated by a redefinition of the z axis. Thus, only the quadratic part of P is irreducible. This part, in general, includes a real and imaginary term. The real term corresponds to an aperture (see Sec. IV) and the imaginary to a focusing element (see Sec. II C). No further analytical elements are needed or, indeed, possible, at this level of approximation. The formalism is, therefore, complete. Of course, one may consider periodic (or even aperiodic) stacks of the elements which have already been discussed, thus covering the topic of diffraction gratings. These though, being constructed of the already defined building blocks, are not inherently new elements.

Note that the diffuser (see Sec. II D) is not included in the discussion above. This is a nonanalytical element which serves to make the connection from the coherent to the incoherent representation, bridging over the partially coherent region

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. CHE-9522232, and by a State of Illinois IBHE grant.

APPENDIX A: PROPERTIES OF THE KIRCHHOFF PROPAGATOR

Given the amplitude of a scalar wave on a suitable surface *S*, the amplitude of the wave in an arbitrary point $\vec{r}$ can be evaluated using the Kirchhoff diffraction integral [1,3]. The well known result is

$$\psi(\vec{r})=\int \frac{-ik(\cos\alpha+\cos\beta)}{4\pi|\vec{r}-\vec{r}\,'|}\exp(ik|\vec{r}-\vec{r}\,'|)\,\psi(\vec{r}\,')\,d\vec{S}, \tag{A1}$$

where *k* is the magnitude of the wave vector, $\vec{r}\,'$ is a point on the surface *S*, and $(\cos\alpha+\cos\beta)$ is the inclination factor. This general result can be easily adapted for the case of paraxial propagation. It is convenient, in such case, to separate longitudinal and transverse coordinates as $\vec{r}=(\vec{\rho},z)$, where *z* is the general direction of the beam propagation and $\vec{\rho}=(x,y)$ is orthogonal to this direction. The paraxial approximation can be summed up by the condition $|\vec{\rho}-\vec{\rho}\,'|\ll|z-z'|$, where $(\vec{\rho},z)$ and $(\vec{\rho}\,',z')$ are points along the wave front propagation path. This condition implies that the angles appearing in the inclination factor are small, thus $(\cos\alpha+\cos\beta)\approx 2$, and justifies the approximation

$$|\vec{r}-\vec{r}\,'|\approx z-z'+\frac{|\vec{\rho}-\vec{\rho}\,'|^2}{2(z-z')}. \tag{A2}$$

The second term in the approximation (A2) may be omitted in the denominator of Eq. (A1), where its contribution is negligible. However, when multiplied by *k*, this term may still contribute significantly to the phase of the exponent in Eq. (A1), thus it needs to be maintained there. Performing the appropriate substitutions while assuming (which can be done without loss of generality) that the surface *S* is orthogonal to *z* (thus $d\vec{S}=d\vec{\rho}\,'$), Eq. (A1) can be rewritten as

$$\psi(\vec{r})=\psi(\vec{\rho},z)=\int \psi(\vec{\rho}\,',z')\,\chi(\vec{\rho}-\vec{\rho}\,',z-z')\,d\vec{\rho}\,', \tag{A3}$$

where the propagator χ is explicitly given by

$$\chi(\vec{r}-\vec{r}\,')=\chi(\vec{\rho}-\vec{\rho}\,',z-z')=\frac{-ik}{2\pi(z-z')}\times\exp\left[ik\left((z-z')+\frac{|\vec{\rho}-\vec{\rho}\,'|^2}{2(z-z')}\right)\right]. \tag{A4}$$

It can be immediately verified, by inspection, that $\chi^*(\vec{r})=\chi(-\vec{r})$, thus complex conjugation of χ is equivalent to reversal of propagation direction. In the specific case of $z'=z$, Eq. (A4) yields singularity and χ is undefined. However the limit of χ for $z'\to z$ can be defined. It is easy to show, through direct integration, that, for $z'\neq z$, χ fulfills the relation

$$\int \chi(\vec{\rho}-\vec{\rho}\,'',z-z'')\,\chi(\vec{\rho}\,''-\vec{\rho}\,',z''-z')\,d\vec{\rho}\,''=\chi(\vec{\rho}-\vec{\rho}\,',z-z'). \tag{A5}$$

On the other hand, substituting $z'=z$ into the left-hand side of Eq. (A5) and integrating yields the result

$$\int \chi(\vec{\rho}-\vec{\rho}\,'',z-z'')\,\chi(\vec{\rho}\,''-\vec{\rho}\,',z''-z)\,d\vec{\rho}\,''=\delta(\vec{\rho}-\vec{\rho}\,'), \tag{A6}$$

where δ is a Dirac delta function. Comparison of Eqs. (A5) and (A6) yields

$$\lim_{z\to z'}\chi(\vec{\rho}-\vec{\rho}\,',z-z')=\delta(\vec{\rho}-\vec{\rho}\,'). \tag{A7}$$

It is possible, using the results above, to describe the general propagation process, where a wave front emitted from a source located at $z=z_0$ interacts with a series of optical elements located at $z_1,\dots,z_n$ along the propagation axis. Assuming that the action of the element located at z_m (for $m=1,\dots,n$) amounts to multiplication of the wave amplitude by $U_m(\vec{\rho}_m)$, an iteration of Eq. (A3) yields the result

$$\psi(\vec{\rho},z)=\int \psi(\vec{\rho}_0,z_0)\,\chi(\vec{\rho}_1-\vec{\rho}_0,z_1-z_0)\,U_1(\vec{\rho}_1)\cdots\times\chi(\vec{\rho}_n-\vec{\rho}_{n-1},z_n-z_{n-1})\,U_n(\vec{\rho}_n)\times\chi(\vec{\rho}-\vec{\rho}_n,z-z_n)\,d\vec{\rho}_0\cdots d\vec{\rho}_n. \tag{A8}$$

The similarity between Eq. (A8) and quantum-mechanical path-integral formalism is noteworthy.

The formalism described so far applies to the most general paraxial situation, where the wave front at any specific location *z* depends on both transverse coordinates. This paper, though, deals with the ‘‘reduced’’ case, where the original wave and all the optical elements vary in only one transverse dimension (which will be taken as *x*, for definiteness) and the other transverse dimension can be integrated away. Introducing the notation $\psi_z(x)=\psi(x,z)$ and performing the *y* integration in Eq. (A3), with χ given by Eq. (A4), the wave propagation in this (transverse, 1D) case is found to be described by

$$\psi_z(x)=\int \psi_0(x')\,\chi(x-x',z)\,dx', \tag{A9}$$

where the ‘‘reduced’’ propagator is given by

$$\chi(x-x',z)=\left(\frac{-ik}{2\pi z}\right)^{1/2}\exp\left[ik\left(z+\frac{|x-x'|^2}{2z}\right)\right]. \tag{A10}$$

The wave amplitude can be used to evaluate average values. Assuming that ψ is normalized [i.e., that $\int \psi^*(x)\psi(x)dx=1$], the average value of an operator $O(x)$ is given by

$$\langle O\rangle = \int \psi^*(x)O(x)\psi(x)dx. \tag{A11}$$

Using Eq. (A9), Eq. (A11) can be converted into an average involving the source amplitude ψ_0 as

$$\begin{aligned}\langle O\rangle &= \int \psi_0^*(x'')\psi_0(x')dx'dx''\int \chi^*(x-x'',z)O(x)\\ &\quad\times\chi(x-x',z)dx\\ &= \int \psi_0^*(x'')\{O\}\psi_0(x')dx'dx'',\end{aligned} \tag{A12}$$

where $\{O\}$ is the result of preaveraging $O(x)$ with the propagators alone. It is convenient, for the purpose of the integration, to replace x',x'' with the variables u,ν defined through

$$x' = u-\nu/2, \quad x''=u+\nu/2. \tag{A13}$$

It can be immediately verified that the Jacobian of this transformation is 1. Using the new variables we get

$$\begin{aligned}&\chi(x-x',z)\chi^*(x-x'',z)\\ &\quad= \frac{k}{2\pi z}\exp\left[\frac{ik(x''-x')}{z}\left(x-\frac{x'-x''}{2}\right)\right]\\ &\quad= \frac{k}{2\pi z}\exp\left(\frac{ik\nu(x-u)}{z}\right).\end{aligned} \tag{A14}$$

Multiplying (A14) by x^n and integrating over x, it can be shown that

$$\{x^n\} = \left(u+\frac{z}{ik}\frac{d}{d\nu}\right)^n\delta(\nu), \tag{A15}$$

where δ is the Dirac delta function and the delta derivatives are used in the standard mathematical meaning. Especially, we have $\{1\}=\delta(\nu)$. Other than 1, the most commonly occurring operators are x^2, θ^2, and $(x\theta+\theta x)$, where θ is, up to a constant, a derivative with respect to x [see Eq. (2.2)]. The averages of these operators involve integrals of the form (A15). With some algebra we get

$$\begin{aligned}\{x^2\} &= u^2\delta(\nu)+\frac{2zu}{ik}\delta'(\nu)-\frac{z^2}{k^2}\delta''(\nu),\\ \left\{\frac{1}{2}(x\theta+\theta x)\right\} &= \frac{u}{ik}\delta'(\nu)-\frac{z}{k^2}\delta''(\nu),\\ \{\theta^2\} &= -\frac{1}{k^2}\delta''(\nu).\end{aligned} \tag{A16}$$

At times it may be convenient to include additional factors, beyond the propagators, in the preaveraging. When dealing with the Gaussian aperture case the wave function ψ is multiplied by an ''aperture amplitude'' of $\exp(-x^2/4d^2)$. Thus all the partial averages are of the form

$$\{O(x)\}_d = \left\{\exp\left(-\frac{x^2}{4d^2}\right)O(x)\exp\left(-\frac{x^2}{4d^2}\right)\right\}. \tag{A17}$$

It should be noted that the aperture amplitude changes the normalization of the wave function. Therefore, assuming the original ψ was normalized, averages of the form (A17) should be multiplied by a normalization factor of the form

$$\begin{aligned}N &= \left[\int \psi_0^*(x'')\{1\}_d\psi_0(x')dx'dx''\right]^{-1}\\ &= \left(\int \psi_0^*(x'')\left\{\exp\left(-\frac{x^2}{2d^2}\right)\right\}\psi_0(x')dx'dx''\right)^{-1}.\end{aligned} \tag{A18}$$

In the following N will be ignored, with the understanding that it is to be included when the final averages are calculated. As before, the main interest is in the averages of the quadratic quantities x^2, θ^2, and $(x\theta+\theta x)$. The results are

$$\begin{aligned}\{x^2\}_d &= d^2\left(1-\frac{k^2d^2\nu^2}{z^2}\right)\{1\}_d,\\ \left\{\frac{1}{2}(x\theta+\theta x)\right\}_d &= \frac{d^2}{z}\left(1-\frac{iku\nu}{z}-\frac{k^2d^2\nu^2}{z^2}\right)\{1\}_d,\\ \{\theta^2\}_d &= \left[\frac{1}{4k^2d^2}+\frac{d^2}{z^2}+\frac{1}{z^2}\left(u-\frac{ik\,d^2\nu}{z}\right)^2\right]\{1\}_d,\end{aligned} \tag{A19}$$

where

$$\{1\}_d = \frac{kd}{\sqrt{2\pi}z}\exp\left[-\left(\frac{k^2d^2\nu^2}{2z^2}+\frac{iku\nu}{z}\right)\right]. \tag{A20}$$

If the initial wave is given in the angle representation, the results are simpler. The Fourier transform of χ from Eq. (A10) is

$$\begin{aligned}\tilde{\chi}(\theta,z) &= \int \chi(w,z)\exp(-ik\theta w)dw\\ &= \left(\frac{-ik}{2\pi z}\right)^{1/2}\int \exp\left[ik\left(z+\frac{w^2}{2z}-\theta w\right)\right]dw\\ &= \exp\left[ikz\left(1-\frac{\theta^2}{2}\right)\right].\end{aligned} \tag{A21}$$

Now, Eq. (2.1) yields

$$\begin{aligned}\tilde{\psi}_z(\theta) &= \sqrt{\frac{k}{2\pi}}\int \psi_z(x)\exp(-ik\theta x)dx\\ &= \sqrt{\frac{k}{2\pi}}\int \psi_0(x')\exp(-ik\theta x')dx'\\ &\quad\times\int \chi(x-x',z)\exp[-ik\theta(x-x')]dx\\ &= \exp\left[ikz\left(1-\frac{\theta^2}{2}\right)\right]\tilde{\psi}_0(\theta) = \tilde{\chi}(\theta,z)\tilde{\psi}_0(\theta).\end{aligned} \tag{A22}$$

Therefore the propagation is represented by a simple multiplication and calculations of averages are done using the source amplitudes directly, with no intermediate integration. At times, though, it is convenient to use a mixed representation, where $\tilde{\psi}_z(\theta)$ is represented in terms of $\psi_0(x)$. From Eq. (A22)

$$\begin{aligned}\tilde{\psi}_z(\theta)&=\exp\left[ikz\left(1-\frac{\theta^2}{2}\right)\right]\tilde{\psi}_0(\theta)\\&=\sqrt{\frac{k}{2\pi}}\int\exp\left\{ik\left[z\left(1-\frac{\theta^2}{2}\right)-x\theta\right]\right\}\psi_0(x)dx\\&=\int\zeta(x,\theta,z)\psi_0(x)dx,\end{aligned}\tag{A23}$$

where the ''mixed'' propagator $\zeta(x,\theta,z)$ is identified as

$$\zeta(x,\theta,z)=\sqrt{\frac{k}{2\pi}}\exp\left\{ik\left[z\left(1-\frac{\theta^2}{2}\right)-x\theta\right]\right\}.\tag{A24}$$

Using the mixed representation, averages are calculated through

$$\begin{aligned}\langle O\rangle&=\int\psi_0^*(x'')\psi_0(x')dx'dx''\int\zeta^*(x'',\theta,z)\\&\quad\times O(\theta)\zeta(x',\theta,z)d\theta\\&=\int\psi_0^*(x'')\{O\}\psi_0(x')dx'dx'',\end{aligned}\tag{A25}$$

where $\{O\}$ is now a partial average over θ. It is easy to show, using the variables u,v from Eq. (A13), that

$$\zeta(x',\theta,z)\zeta^*(x'',\theta,z)=\frac{k}{2\pi}\exp(ik\theta\nu).\tag{A26}$$

Multiplying Eq. (A26) by θ^n and integrating over θ yields

$$\{\theta^n\}=\left(\frac{1}{ik}\frac{d}{d\nu}\right)^n\delta(\nu),\tag{A27}$$

similar to Eq. (A15). Especially we have $\{1\}=\delta(\nu)$.

Equations (A24) and (A25) can be used to evaluate the averages of x^2, θ^2, and $(x\theta+\theta x)$, replicating the results of Eq. (A16). More interesting, though, is the case of a Gaussian ''momentum aperture,'' the effect of which is to multiply the angle amplitude by $\exp(-\theta^2/4\delta^2)$, similar to the position aperture discussed above. Thus, similar to Eq. (A17), the partial averages are of the form

$$\{O(\theta)\}_\delta=\left\{\exp\left(-\frac{\theta^2}{4\delta^2}\right)O(\theta)\exp\left(-\frac{\theta^2}{4\delta^2}\right)\right\}.\tag{A28}$$

Up to a normalization factor [as in Eq. (A18), with $\{1\}_\delta$ replacing $\{1\}_d$] which, again, is needed here, the averages of the quadratic factors are now

$$\begin{aligned}\{x^2\}_\delta&=\left(\frac{1}{4k^2\delta^2}+\delta^2z^2+(u+ik\delta^2z\nu)^2\right)\{1\}_\delta,\\\left\{\frac{1}{2}(x\theta+\theta x)\right\}_\delta&=\delta^2z\left(1+\frac{iku\nu}{z}-k^2\delta^2\nu^2\right)\{1\}_\delta,\\\{\theta^2\}_\delta&=\delta^2(1-k^2\delta^2\nu^2)\{1\}_\delta,\end{aligned}\tag{A29}$$

where

$$\{1\}_\delta=\frac{k\delta}{\sqrt{2\pi}}\exp\left[-\left(\frac{k^2\delta^2\nu^2}{2}\right)\right].\tag{A30}$$

APPENDIX B: GEOMETRICAL PHASE SPACE APPROACH TO BEAM PROPAGATION

The discussion of beam propagation in this appendix is purely geometrical. Wave properties are ignored and the beam is represented by an intensity distribution in one transverse dimension (thus a 2D phase space) $I_z(x,\theta)$, which propagates along the z axis. For the sake of convenience I_z is assumed to be normalized, i.e., $\int I_z(x,\theta)dx\,d\theta=1$. Using the notation

$$\vec{\rho}=\begin{pmatrix}x\\\theta\end{pmatrix},\tag{B1}$$

for an individual phase space point, the distribution I_z, for any specific value of z, can be characterized by the matrix

$$\mathbf{V}=\langle\vec{\rho}\vec{\rho}^T\rangle=\begin{pmatrix}\langle x^2\rangle&\langle x\theta\rangle\\\langle\theta x\rangle&\langle\theta^2\rangle\end{pmatrix}=\begin{pmatrix}V_{xx}&V_{x\theta}\\V_{\theta x}&V_{\theta\theta}\end{pmatrix},\tag{B2}$$

where

$$\begin{aligned}\langle x^2\rangle&=\int I_z(x,\theta)x^2dx\,d\theta,\\\langle\theta^2\rangle&=\int I_z(x,\theta)\theta^2dx\,d\theta,\\\langle x\theta\rangle=\langle\theta x\rangle&=\int I_z(x,\theta)x\theta\,dx\,d\theta.\end{aligned}\tag{B3}$$

In the paraxial approximation $\vec{\rho}$ transforms linearly under the action of propagation and/or optical elements, i.e., $\vec{\rho}'=\mathbf{T}\vec{\rho}$ where $\mathbf{T}$ is a linear operator. According to the definition (B2), the resulting transformation of $\mathbf{V}$ is $\mathbf{V}'=\mathbf{TVT}^T$. Since, as a consequence of Liouville's theorem, the determinant of $\mathbf{V}$ remains invariant under all such transformations, the operator $\mathbf{T}$ must fulfill $|\mathbf{T}|=\pm1$. The invariance of $|\mathbf{V}|$ allows us to define the invariant emittance of the beam ε by

$$\varepsilon^2=|\mathbf{V}|=\langle x^2\rangle\langle\theta^2\rangle-\langle x\theta\rangle^2.\tag{B4}$$

Of special interest are locations along the beam axis z where $\mathbf{V}$ is diagonal. Such locations will be referred to as ''sources.'' In the following it will be shown that any beam has a ''source.'' At the source location

$$\mathbf{V}=\begin{pmatrix}\sigma_x^2 & 0\\ 0 & \sigma_\theta^2\end{pmatrix} \tag{B5}$$

and

$$\varepsilon^2=\sigma_x^2\sigma_\theta^2. \tag{B6}$$

But, as ε is invariant, Eq. (B6) viewed as an expression of ε in terms of the source's spatial and angular dimensions is valid everywhere, thus the relation $\varepsilon=\sigma_x\sigma_\theta$ can be used as an alternative definition of emittance.

The operator corresponding to a propagation over a distance z is

$$\mathbf{T}_z=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}. \tag{B7}$$

Note that, indeed, $|\mathbf{T}|=1$. $\mathbf{V}$, in this case, transforms according to

$$\mathbf{V}'=\mathbf{T}_z\mathbf{V}\mathbf{T}_z^T=\begin{pmatrix}V_{xx}+z(V_{x\theta}+V_{\theta x})+z^2V_{\theta\theta} & V_{x\theta}+zV_{\theta\theta}\\ V_{\theta x}+zV_{\theta\theta} & V_{\theta\theta}\end{pmatrix}. \tag{B8}$$

Especially, choosing

$$z=-z_s=-\frac{V_{x\theta}}{V_{\theta\theta}} \tag{B9}$$

yields

$$\mathbf{V}'=\begin{pmatrix}V_{xx}-\dfrac{V_{x\theta}^2}{V_{\theta\theta}} & 0\\ 0 & V_{\theta\theta}\end{pmatrix}=\begin{pmatrix}\dfrac{\varepsilon^2}{V_{\theta\theta}} & 0\\ 0 & V_{\theta\theta}\end{pmatrix}. \tag{B10}$$

Therefore, a beam with an arbitrary $\mathbf{V}$ can be described as propagating over distance z_s, from a source with sizes σ_x,σ_θ, given by

$$\sigma_x^2=\frac{\varepsilon^2}{V_{\theta\theta}}=V_{xx}-\frac{V_{x\theta}^2}{V_{\theta\theta}},\qquad \sigma_\theta^2=V_{\theta\theta}. \tag{B11}$$

Focusing of the beam can be similarly described. A device with focal length F is represented by the operator

$$\mathbf{T}_F=\begin{pmatrix}1 & 0\\ -\dfrac{1}{F} & 1\end{pmatrix}, \tag{B12}$$

yielding

$$\mathbf{V}'=\mathbf{T}_F\mathbf{V}\mathbf{T}_F^T=\begin{pmatrix}V_{xx} & V_{x\theta}-\dfrac{1}{F}V_{xx}\\ V_{\theta x}-\dfrac{1}{F}V_{xx} & V_{\theta\theta}-\dfrac{1}{F}(V_{x\theta}+V_{\theta x})+\dfrac{1}{F^2}V_{xx}\end{pmatrix}. \tag{B13}$$

This transformation changes the original beam to one corresponding to a new source with different source parameters $\sigma_x',\sigma_\theta',z_s'$. The new parameters can be found using Eqs. (B9) and (B11). Expressed in terms of the original parameters the results are

$$\sigma'^2_x=\frac{F^2\sigma_x^2\sigma_\theta^2}{\sigma_x^2+(F-z_s)^2\sigma_\theta^2},$$
$$\sigma'^2_\theta=\frac{\sigma_x^2+(F-z_s)^2\sigma_\theta^2}{F^2}, \tag{B14}$$
$$z_s'=\frac{F[z_s(F-z_s)\sigma_\theta^2-\sigma_x^2]}{\sigma_x^2+(F-z_s)^2\sigma_\theta^2}.$$

The effect of apertures is easiest to analyze for Gaussian beams. These are beams with a (normalized) intensity distribution given by

$$I(\vec{\rho})=\frac{1}{2\pi\varepsilon}\exp\left(-\frac{1}{2}\vec{\rho}^T\mathbf{Q}\vec{\rho}\right), \tag{B15}$$

where

$$\mathbf{Q}=\mathbf{V}^{-1}=\frac{1}{\varepsilon^2}\begin{pmatrix}\langle\theta^2\rangle & -\langle x\theta\rangle\\ -\langle\theta x\rangle & \langle x^2\rangle\end{pmatrix}. \tag{B16}$$

Now, the effect of a general Gaussian aperture can be represented by

$$I'(\vec{\rho})=I(\vec{\rho})\exp\left(-\frac{1}{2}\vec{\rho}^T\mathbf{S}\vec{\rho}\right), \tag{B17}$$

where $\mathbf{S}$ is a symmetric, semipositive operator. For beams of the form (B15) the action of the aperture amounts to the substitution

$$\mathbf{Q}'=\mathbf{Q}+\mathbf{S}. \tag{B18}$$

Based on the fact that both $\mathbf{Q}$ and $\mathbf{S}$ are positive operators, it is easy to show that an aperture reduces the emittance as

$$\frac{1}{\varepsilon'^2}=|\mathbf{Q}'|\geqslant|\mathbf{Q}|=\frac{1}{\varepsilon^2}. \tag{B19}$$

The basic apertures are position and angle (momentum) apertures. A position aperture is represented by

$$\mathbf{S}=\begin{pmatrix}\dfrac{1}{d^2} & 0\\ 0 & 0\end{pmatrix}, \tag{B20}$$

generating the new $\mathbf{Q}$ and $\mathbf{V}$ operators

$$\mathbf{Q}' = \frac{1}{\varepsilon^2}\begin{pmatrix} \langle \theta^2\rangle + \dfrac{\varepsilon^2}{d^2} & -\langle x\theta\rangle \\ -\langle \theta x\rangle & \langle x^2\rangle \end{pmatrix} \rightarrow \mathbf{V}'$$

$$= \frac{d^2}{d^2+\langle x^2\rangle}\begin{pmatrix} \langle x^2\rangle & \langle x\theta\rangle \\ \langle \theta x\rangle & \langle \theta^2\rangle + \dfrac{\varepsilon^2}{d^2} \end{pmatrix}. \qquad \text{(B21)}$$

The new emittance is

$$\varepsilon'^2 = \frac{d^2\varepsilon^2}{d^2+\langle x^2\rangle}. \qquad \text{(B22)}$$

The angle (momentum) aperture calculation is similar, with an $1/\delta^2$ term at the $(\theta\theta)$ location of **S** instead of the $1/d^2$ term at the (xx) location [see Eq. (B20)]. The new **Q** and **V** in this case are

$$\mathbf{Q}' = \frac{1}{\varepsilon^2}\begin{pmatrix} \langle \theta^2\rangle & -\langle x\theta\rangle \\ -\langle \theta x\rangle & \langle x^2\rangle + \dfrac{\varepsilon^2}{\delta^2} \end{pmatrix} \rightarrow \mathbf{V}'$$

$$= \frac{\delta^2}{\delta^2+\langle \theta^2\rangle}\begin{pmatrix} \langle x^2\rangle + \dfrac{\varepsilon^2}{\delta^2} & \langle x\theta\rangle \\ \langle \theta x\rangle & \langle \theta^2\rangle \end{pmatrix}, \qquad \text{(B23)}$$

and the new emittance is

$$\varepsilon'^2 = \frac{\delta^2\varepsilon^2}{\delta^2+\langle \theta^2\rangle}. \qquad \text{(B24)}$$

The modified source parameters generated by both types of apertures are listed in Table II.

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